Int J Theor Phys (2010) 49: 1781–1787
DOI 10.1007/s10773-010-0359-5

Hawking Radiation from Charged Kerr Black Hole Beyond Semi-classical Approximation

Gao Jie · Luo Zhi-Quan · Liu Wei-Wei · Li Gang

Received: 17 December 2009 / Accepted: 13 April 2010 / Published online: 22 April 2010

Abstract The Hawking radiation from charged Kerr black hole via the method beyond semi-classical approximation is studied. In our work, we apply the WKB approximation method and the quantum tunneling method, then calculate the tunneling rate and further correct Hawking entropy to charged Kerr black hole. It is shown that the result is still in agreement with the unitary theory, the entropy of the black hole contains three parts: the usual Bekenstein-Hawking entropy, the logarithmic term and the inverse area term. Apart from coefficients, our correction to the charged Kerr black hole entropy is consistent with results of loop quantum gravity.

Keywords Charged Kerr black hole · Hawking radiation · Corrected entropy

1 Introduction

Black hole, is one of objects in universe, although it isn't proved directly nowadays, it exists truly. So it is very interesting to investigate the black hole. Since 1974, Hawking has made a remarkable discovery that a black hole can radiate thermally. Hawking radiation has attracted many people's attention and many methods have been brought forward to derive it [1–17]. Above all, the most simpliest method is the semi-classical quantum approach modeling Hawking radiation as tunneling effect, which was proposed by Kraus and Wilczek, and was also developed by a lot of researchers [13–26]; and on basis of them, Ryan Kerner and R.B. Mann et al. proposed Hawking radiation from black hole beyond semi-classical approximation, making the method further investigated and developed. Hawking radiation is radial result from quantum effect of black hole, we can obtain black hole tunneling rate and Hawking temperature. After this method succeeded, many people also probed tunneling rate and Hawking temperature of black holes. Ryan Kerner and R.B. Mann investigated fermions tunneling of rotating and charged Kerr-Newman black hole Hawking radiation; Chen

Supported by National Natural Science Foundation (No. 10778719).

G. Jie (✉) · L. Zhi-Quan · L. Wei-Wei · L. Gang
Institute of Theoretical Physics, China West Normal University, Nanchong 637002, China
e-mail: gaolibang007@163.com

et al probed tunneling radiation of charged dilatonic black hole at de Sitter horizon; Zhang studied spherically symmetric black hole quantum tunneling and black entropy correction; Li et al. discussed tunneling radiation from BTZ black hole and Kerr black hole; Jiang et al researched on fermions tunneling from 5-dimensional black hole.

These results above help to develop black hole physics. However, in these researches, they omitted charged Kerr black hole quantum tunneling and its entropy correction. Because many tunneling theories based on semi-classical approximation, what people can obtain is just approximate conclusions, the effect of some terms which have $\hbar$ as multiplicator has been ignored in solving the Hamilton-Jacobi equation. In order to solve this problem, Banerjee and Majhi put forward a new method to research the black hole tunneling radiation beyond the semi-classical approximation in 2008, they finally obtained corrected Hawking temperature and corrected black hole entropy. Zhang obtained corrected black hole entropy using quantum tunneling method; Lin et al investigated fermions tunneling from charged static Reissner-Nordstrom, then got corrected Hawking temperature and corrected black hole entropy, and they further developed Banerjee and Majhi's theory; However, in these researches, they pay little attention to corrected entropy problem about charged Kerr black hole. In our work, we further study charged Kerr black hole, and obtain corrected entropy of charged Kerr black hole. In other words, we investigate axial symmetric charged Kerr black hole tunneling beyond semi-classical approximation, and we finally obtain tunneling rate and corrected entropy of charged Kerr black hole. This work makes the theory about black hole more complete and more harmonious.

2 Charged Kerr Black Hole Tunneling and Corrected Entropy

Charged Kerr black hole metric in Boyer-Lindquist coordinate system can be expressed as [3]:

$$ds^2 = d\bar{t}^2 - \frac{2Mr - Q^2}{\Sigma}\left(d\bar{t} - a\sin^2\theta d\bar{\phi}\right)^2 - \frac{\Sigma}{\Delta}dr^2 - \Sigma d\theta^2 - \left(r^2 + a^2\right)\sin^2\theta d\bar{\phi}^2 \quad (1)$$

where $\Sigma = r^2 + a^2\cos^2\theta$, $\Delta = r^2 + a^2 + Q^2 - 2Mr$.

The specific angular momentum $a = J/M$ is kept as a constant through this paper. Via Painleve-type coordinate transformation and dragging coordinate transformation, we can obtained the desired 3-dimensional dragged Painleve-Kerr-Newman metric as follows.

A generalized Painleve-type coordinate transformation:

$$d\bar{t} = dt - \frac{\sqrt{(2Mr - Q^2)(r^2 + a^2)}}{\Delta}dr \quad (2)$$

$$d\bar{\phi} = d\phi - \frac{a}{\Delta}\sqrt{\frac{2Mr - Q^2}{r^2 + a^2}}dr \quad (3)$$

The dragging coordinate transformation:

$$d\phi = \frac{a(r^2 + a^2 - \Delta)}{(r^2 + a^2)^2 - \Delta a^2\sin^2\theta}dt \quad (4)$$

According to (1)–(4), we can obtained the desired 3-dimensional dragged Painleve-Kerr-Newman metric as follows:

$$d\hat{s}^2 = \frac{\Delta\Sigma}{(r^2+a^2)^2 - \Delta a^2 \sin^2\theta} dt^2 - \frac{\Sigma}{r^2+a^2} dr^2 - 2\frac{\sqrt{(2Mr-Q^2)(r^2+a^2)}\Sigma}{(r^2+a^2)^2 - \Delta a^2 \sin^2\theta} dt dr - \Sigma d\theta^2 \quad (5)$$

According to de Broglie's hypothesis and the definition of the phase (group) velocity, the outgoing particle that can be considered as a massive shell corresponding to a kind of 's-wave'.

After coordinate transformation, we finally obtain charged geodesic in the radial direction [3]:

$$\dot{r} = \frac{dr}{dt} = -\frac{g_{tt}}{2g_{tr}} = \frac{\Delta}{2(r^2+a^2)\sqrt{1-\frac{\Delta}{r^2+a^2}}} \quad (6)$$

where $\Delta = r^2 + a^2 + Q^2 - 2Mr$. Note: to include the particle's self-interaction effect after the charged particle emission, the mass and charged parameters in (5) and (6) should be replaced M and q with $M - \omega$ and $Q - q$ to describe the motion of the particle correctly.

In the following discussion, we further consider tunneling process of a radiation particle. Parikh and Wilczek applied the WKB approximation to calculate the emission rate of a tunneling particle (s-shell). Applying Zhang's method [27], we investigate the tunneling process of a massive particle, starting with the WKB method and the barrier penetration. Schrodinger's equation for the motion of a particle in a centrally symmetric field is given by:

$$\nabla\Psi + (2m/\hbar^2)(E - U(r))\Psi = 0 \quad (7)$$

Considering the following radial equation:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \frac{l(l+1)}{r^2}R + \frac{2m}{\hbar^2}(E-U(r))R = 0 \quad (8)$$

By the substitution:

$$R(r) = X(r)/r \quad (9)$$

equation (8) is equal to the form:

$$\frac{d^2X}{dr^2} + \left[\frac{2m}{\hbar^2}(E-U(r)) - \frac{l(l+1)}{r^2}\right]X = 0 \quad (10)$$

For s-wave, $1 = 0$, the equation $X(r)$ is:

$$\frac{d^2X}{dr^2} + \left[\frac{2m}{\hbar^2}(E-U(r))\right]X = 0 \quad (11)$$

Considering the self-gravitation reliably the tunneling particle as a spherical shell, in the Parikh- Wilczek framework. In this way, when it emits from the black hole, the matter gravity system transits from one spherical state to another, the de Broglie wave function of emission spherical shell is:

$$\Psi(r) = X(r)/r \quad (12)$$

The WKB wave function of a particle can be written as follows:

$$\Psi(r) = X(r)/r = \frac{1}{r}\exp\left|\frac{iS(r)}{\hbar}\right| \tag{13}$$

where

$$S(r) = S_0(r) + \left(\frac{\hbar}{i}\right)S_1(r) + \left(\frac{\hbar}{i}\right)^2 S_2(r) + \cdots \tag{14}$$

Substituting (13) into (11) yields:

$$S_0 = \pm\int p_r dr \tag{15}$$

$$2S_0'S_1' + S_0'' = 0 \tag{16}$$

$$2S_0'S_2' + (S_1')^2 + S_1'' = 0 \tag{17}$$

where a prime denotes differentiation with respect to r.

To evaluate the probability of a particle passing through the barrier, we divide the whole region of motion of the particle by two tunnelling points A and B into three parts: ingoing and reflecting region, barrier region and the outgoing region. The particle can move freely in ingoing and outgoing, but barrier region is classically inaccessible. The WKB wave function is from reference [27].

The probability of barrier penetration is:

$$\Gamma_p = \frac{j_{out}}{j_{in}} = \frac{v|\Psi_{out}|^2}{v|\Psi_{in}|^2} = \frac{v(X_{out}(b)/b)^2}{v(X_{in}(a)/a)^2} = \frac{a^2}{b^2}\exp\left[-\frac{2\mathrm{Im}S_0}{\hbar}\right] \tag{18}$$

Let's calculate the phase space factor corresponding to the charged Kerr black hole tunneling.

For charged Kerr black hole, we can easily obtained the canonical momentum p_r and the imaginary part of the action $\mathrm{Im}S_0$:

$$p_r = \int_0^{p_r} dp_r' = \int \frac{dH}{\dot{r}} = -i\pi\cdot\frac{r^2+a^2}{r} \tag{19}$$

where $\dot{r} = \frac{dr}{dt} = -\frac{g_{tt}}{2g_{tr}} = \frac{\Delta}{2(r^2+a^2)\sqrt{1-\frac{\Delta}{r^2+a^2}}}$

$$\mathrm{Im}S_0 = \int_{r_i}^{r_f} p_r dr = -\frac{1}{8}(A_f - A_i) - \frac{1}{2}\pi a^2 \ln\frac{A_f}{A_i} \tag{20}$$

The probability of barrier penetration is:

$$\Gamma_p = \frac{r_i^2}{r_f^2}\exp\left[-\frac{2\mathrm{Im}S_0}{\hbar}\right] = \exp\left\{\left[\frac{A_f}{4\hbar} + \left(\frac{\pi a^2}{\hbar} - 1\right)\ln A_f\right] - \left[\frac{A_i}{4\hbar} + \left(\frac{\pi a^2}{\hbar} - 1\right)\ln A_i\right]\right\} \tag{21}$$

In this paper, we investigate the transition of the matter-gravity system from one spherical state to another at the same energy. The transition corresponds to the production and barrier penetration of the massive spherical shell. In other words, this process contains two stages. The first stage is the production of the spherical shell from the vacuum fluctuation near the event horizon. The second stage is the barrier penetration. So, the rate of transition from the initial spherical state to the final spherical state is:

$$\Gamma(i \to f) = \Gamma_c \cdot \Gamma_p = \Gamma_c \cdot \exp\left\{\left[\frac{A_f}{4\hbar} + \left(\frac{\pi a^2}{\hbar} - 1\right)\ln A_f\right] - \left[\frac{A_i}{4\hbar} + \left(\frac{\pi a^2}{\hbar} - 1\right)\ln A_I\right]\right\}$$
$$= \Gamma_c \cdot \left[\exp\left(\frac{A_f}{4\hbar} + \alpha_0 \ln \frac{A_f}{4\hbar}\right) - \left(\frac{A_i}{4\hbar} + \alpha_0 \ln \frac{A_i}{4\hbar}\right)\right] \tag{22}$$

where $\alpha_0 = \frac{\pi a^2}{\hbar} - 1$ is a constant.

Let's compare (22) with the unitary result in Quantum Mechanics, $\Gamma(i \to f) = |M_{fi}|^2$. (phase space factor, detailed process is from reference [27]. $|M_{fi}|^2$ is the probability amplitude of the process, this case is related to the process of the first stage. Thus, we have

$$\text{Phase space factor} = \frac{N_f}{N_i} = \frac{e^{S_f}}{e^{S_i}} = e^{S_f - S_i} \tag{23}$$

we naturally obtain the expression of the charged Kerr black hole entropy to the first order correction

$$S_q = \frac{A_H}{4\hbar} + \alpha_0 \ln \frac{A_H}{4\hbar} \tag{24}$$

3 Second Order Correction to Charged Kerr Black Hole Entropy

Let's calculate the tunneling rate to the second order approximation. In order to get the second order correction of the black hole entropy, we write the WKB wave function to the second order approximation.

$$X(r) = \exp\left[\frac{iS_0(r)}{\hbar} + S_1(r) + \frac{\hbar}{i}S_2(r)\right] \tag{25}$$

Where

$$S_2 = \int -\frac{(S_1'^2 + S_1'')}{2S_0'}dr \tag{26}$$

It is similar to the treatment in Sect. 2, we can obtain the wave function of ingoing and reflecting region, barrier region and the outgoing region, and get the expression of $S_2(r)$.

The expression of $S_2(r)$ in ingoing and reflecting region is:

$$S_2 = \int_r^A -\frac{(S_1'^2 + S_1'')}{2S_0'}dr \tag{27}$$

The expression of $S_2(r)$ in barrier region is obtained as:

$$S_2 = \int_A^r -\frac{(S_1'^2 + S_1'')}{2S_0'}dr \tag{28}$$

The expression of $S_2(r)$ in the outgoing region is:

$$S_2=\int_A^B -\frac{(S_1'^2+S_1'')}{2S_0'}dr \tag{29}$$

In ingoing and reflecting region, the ingoing flux density is

$$j_{in}=\frac{-i\hbar}{2m}\left(\psi_{in}\frac{\partial}{\partial r}\psi_{in}^*-\psi_{in}^*\frac{\partial}{\partial r}\psi_{in}\right)=v|\psi_{in}^2|=\frac{1}{A^2} \tag{30}$$

In the outgoing region, the outgoing flux density is

$$j_{out}=\frac{-i\hbar}{2m}\left(\psi_{out}\frac{\partial}{\partial r}\psi_{out}^*-\psi_{out}^*\frac{\partial}{\partial r}\psi_{out}\right)=v|\psi_{out}^2|=\frac{1}{B^2}\exp[\mathrm{Im}S_0-\hbar^2\mathrm{Im}S_2] \tag{31}$$

Thus, we have

$$\Gamma=\frac{j_{out}}{j_{in}}=\frac{A^2}{B^2}\exp\left[-\frac{2}{\hbar}\left(\mathrm{Im}S_0-\hbar^2\mathrm{Im}S_2\right)\right] \tag{32}$$

For charged Kerr black hole tunneling, in classically inaccessible region, we have

$$S_0'=p_r=-i\pi\frac{r^2+a^2}{r},\qquad S_0''=-i\pi\frac{r^2-a^2}{r^2}, \tag{33}$$

$$S_1'=-\frac{1}{2}\frac{S_0''}{S_0'}=-\frac{r^2-a^2}{2r(r^2+a^2)},\qquad S_1''=\frac{(r^4-4a^2r^2-a^4)}{2r^2(r^2+a^2)^2} \tag{34}$$

From (29) we can easily obtain

$$S_2'=-\frac{1}{2S_0'}(S_1'^2+S_1'')=-\frac{i}{8\pi}\cdot\frac{3r^4-10a^2r^2-a^4}{r(r^2+a^2)^3} \tag{35}$$

Thus

$$\begin{aligned}S_2=\int_{r_i}^{r_f}S_2'dr=\frac{i}{8\pi}\bigg\{&\left[\frac{5\pi}{A_f+A_0}+\frac{1}{2a^2}\ln\left(1-\frac{A_0}{A_f+A_0}\right)-\frac{12\pi A_0}{(A_f+A_0)^2}\right]\\&-\left[\frac{5\pi}{A_i+A_0}+\frac{1}{2a^2}\ln\left(1-\frac{A_0}{A_i+A_0}\right)-\frac{12\pi A_0}{(A_i+A_0)^2}\right]\bigg\}\end{aligned} \tag{36}$$

where $A_0=4\pi a^2$ is a constant.

Substituting (22), (35) into (32) and considering

$$\Gamma(i\to f)=|M_{fi}|^2\quad\text{(phase space factor)} \tag{37}$$

yield:

phase space factor

$$\begin{aligned}&=\exp\bigg\{\frac{A_f}{4\hbar}+\alpha_0\ln\frac{A_f}{4\hbar}+\frac{\hbar}{4\pi}\left[\frac{5\pi}{A_f+A_0}+\frac{1}{2a^2}\ln\left(1-\frac{A_0}{A_f+A_0}\right)-\frac{12\pi A_0}{(A_f+A_0)^2}\right]\\&\quad-\frac{A_i}{4\hbar}-\alpha_0\ln\frac{A_i}{4\hbar}-\frac{\hbar}{4\pi}\left[\frac{5\pi}{A_i+A_0}+\frac{1}{2a^2}\ln\left(1-\frac{A_0}{A_i+A_0}\right)-\frac{12\pi A_0}{(A_i+A_0)^2}\right]\bigg\}\end{aligned} \tag{38}$$

Comparing (38) with (23), we get the expression of the charged Kerr black hole entropy to the second order correction:

$$S_q = \frac{A_H}{4\hbar} + \alpha_0 \ln \frac{A_H}{4\hbar} + \frac{\hbar}{4\pi}\left[\frac{5\pi}{A_H + A_0} + \frac{1}{2a^2}\ln\left(1 - \frac{A_0}{A_H + A_0} - \frac{12\pi A_0}{(A_H + A_0)^2}\right)\right] \quad (39)$$

where α_0 is constant.

The result is consistent with the unitary theory, but is also in agreement with the general formulation of the black hole entropy. The tunneling rate is

$$\Gamma(i \to f) \sim e^{\Delta S_q} \quad (40)$$

4 Conclusion

Our work shows that the result we obtained is consistent with the unitary theory. The entropy of the black hole will contain three parts: the usual Bekenstein-Hawking entropy, the logarithmic term and inverse area term. In our conclusion, our correction to the black hole entropy is consistent with that in loop quantum gravity, apart from coefficient. The result is also agreement with the general formulation of the black hole entropy. $S_q = \frac{A_H}{4l_p^2} + \alpha \ln \frac{A_H}{4l_p^2} + O(\frac{l_p^2}{A_H}) + \text{const}$. However, that the second order correction to charged Kerr black hole entropy is not easily enough. So, we will study further in another paper. The result is very important to realize the black hole.

References

1. Hawking, S.W.: Nature **248**, 30 (1974)
2. Hawking, S.W.: Commun. Math. Phys. **43**, 199 (1975)
3. Jiang, Q.Q., Wu, S.Q., Cai, X.: Phys. Rev. D **75**, 064029 (2007)
4. Jiang, Q.Q., Wu, S.Q.: Phys. Lett. B **647**, 200 (2007)
5. Lin, K., Yang, S.Z.: Chin. Phys. Lett. **26**, 100401 (2009)
6. Yang, S.Z., Chen, D.Y.: Chin. Phys. Lett. **24**, 1479 (2007)
7. Yang, S.Z., Chen, D.Y.: Chin. Phys. B **17**, 817 (2008)
8. Chen, D.Y., Jiang, Q.Q., Zu, X.T.: Phys. Lett. B **665**, 106 (2008)
9. Chen, D.Y., Jiang, Q.Q., Zu, X.T.: Class. Quantum Gravity **25**, 205022 (2008)
10. Jiang, Q.Q., Wu, S.Q., Cai, X.: Phys. Rev. D **73**, 064003 (2006)
11. Jiang, Q.Q., Wu, S.Q., Cai, X.: Phys. Lett. B **651**, 58 (2007)
12. Jiang, Q.Q., Cai, X.: J. High Energy Phys. **11**, 110 (2009)
13. Yang, S.Z.: Acta Phys. Sin. **53**, 4007 (2004)
14. Zhang, J.Y., Zhao, Z.: Phys. Sin. **55**, 3796 (2006)
15. Zhang, J.Y., Zhao, Z.: Nucl. Phys. B **725**, 173 (2005)
16. Lin, K., Yang, S.Z.: Acta. Phys. Sin. B (2008)
17. Lin, K., Yang, S.Z.: Phys. Lett. B **674**, 127–130 (2009)
18. Lin, K., Yang, S.Z.: Phys. Rev. D **79**, 064035 (2009)
19. Lin, K., Yang, S.Z.: arXiv:0903.1983v1 [gr-qc/]
20. Lin, K., Yang, S.Z.: Acta Phys. Sin. **58**, 744–748 (2009)
21. Banerijee, R., Majhi, B.R.: J. Hight Energy Phys. **0806**, 095 (2008). arXiv:0805.2220
22. Banerijee, R., Majhi, B.R.: Phys. Lett. B **674**, 218 (2009)
23. Banerijee, R., Majhi, B.R.: Phys. Lett. B **675**, 243 (2009). arXiv:0903.0250
24. Banerijee, R., Modak, S.K.: J. Hight Energy Phys. 0905:063 (2009). arXiv:0903.3321
25. Majhi, B.R., Samanta, S.: arXiv:0901.2258
26. Banerijee, R., Majhi, B.R.: Phys. Lett. B **674**, 218–222 (2009). arXiv:0808.3688
27. Zhang, J.Y.: Phys. Lett. B **668**, 353–356 (2008)